Cutting rules and perturbative unitarity of noncommutative electric-type field theories from string theory

Alessandro Torrielli[1]

[1] *Dipartimento di Fisica "G.Galilei", Via Marzolo 8, 35131 Padova, Italy*

INFN, Sezione di Padova, Italy

(torrielli@pd.infn.it)

Abstract

We discuss the breakdown of perturbative unitarity of noncommutative quantum field theories in electric-type background from the point of view of cutting rules. We consider the analytic structure of string loop two-point functions and then perform the zero slope limit of Seiberg and Witten. In this way we pick up how the unphysical tachyonic branch point appears in the effective field theory.

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I. INTRODUCTION

The problem of the perturbative unitarity of noncommutative quantum field theories has been faced since the appearance of these theories as an effective description for open string theory amplitudes in an antisymmetric constant background [1]. The absence of a straightforward hamiltonian formulation for the case of an electric-type noncommutativity (in which the time variable is involved) and the bizarre dynamical features of its scattering amplitudes [2] have soon casted doubts on the consistency of this kind of theories. One of the first confirmations of the breakdown of perturbative unitarity was given in [3] for a noncommutative scalar theory: Cutkoski's rules were found to hold only in the magnetic case; in the electric case an additional tachyonic branch cut is present. An analogous result was found in [4] for gauge theories: here again the cutting rules hold if the noncommutativity does not involve the time variable, otherwise new intermediate tachyonic states call for being inserted in order to explain the analytic structure of the vacuum polarization tensor. These new possible states were considered in [5] in strict connection with the situation one encounters in open string scattering, where unitarity is recovered provided one add the appropriate intermediate closed strings exchanges. This is related to the fact that in the electric case the Seiberg-Witten limit does not succeed in decoupling the closed string sector, as well as the open massive string states [6]. The best one can do in this situation is to set the electric field at its critical value, ending with a theory of only open strings (NCOS), namely decoupling all closed strings but keeping the open massive tower of states.

From the point of view of the behaviour of open strings in electric backgrounds the situation is well understood [7–13]: there is a critical value of the field beyond which the string becomes unstable. In particular an electric field along the string can strecht it and balance its energy, reducing its effective tension to zero, making pair production possible from the vacuum without energy loss. The limit of Seiberg and Witten precisely forces the electric field to overcome this critical value; therefore the corresponding field theory describes an unstable string. There is no way to reach a complete decoupling and thereby an effective noncommutative field theory without overcoming this critical point.

Our aim is to make this situation clear from the point of view of the cutting rules, namely we want to reproduce the result of [3] from string amplitudes, precisely looking at what happens to the branch cuts when one performs the Seiberg-Witten limit. We see that string theory is perfectly unitary below the critical value of the electric field, and we find consistently that the branch cuts in the complex plane are parameterized by a quantity that changes sign when the electric field overcomes its critical value. We clarify how this situation is reflected in the analytical structure of the field theory amplitudes.

II. THE CALCULATION

Let's begin considering one-loop tachyon amplitudes in bosonic open string theory in the presence of a constant B-field living on a Dp-brane. These amplitudes have already been studied [14–16], in particular in [14] it was shown that the noncommutative effective field theory on the brane in the zero-slope limit was precisely NC ϕ^3. This was also one of the cases examined in [3] where the breakdown of cutting rules was found.

We therefore start from writing the string amplitude for a generic dimension p of the brane, governed by the sigma model action

$$S = \frac{1}{4\pi\alpha'} \int_{C_2} d^2z \, (g_{ij}\partial_a X^i \partial^a X^j - 2i\pi\alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j). \tag{1}$$

The string world-sheet is conformally mapped into the cilinder $C_2 = \{0 \leq \Re w \leq 1, w = w + 2i\tau\}$; τ is the modulus of the cilinder, the metric g and the antisymmetric tensor B are constant closed-string sector backgrounds. The indices i and j live on the brane: since outside the brane a constant B can be gauged away, we are interested only in the brane action. The propagator in this situation, with the new boundary conditions imposed by the B-term in (1), has been constructed in [8], [9], [14]. If one sets $w = x + iy$, the relevant propagator on the boundary of the cilinder ($x = 0, 1$) can be written as [14]

$$G(y, y') = \frac{1}{2}\alpha' g^{-1} \log q - 2\alpha' G^{-1} \log \Big[q^{\frac{1}{4}}\, \vartheta_4 (\frac{|y - y'|}{2\tau}, \frac{i}{\tau}) / D(\tau) \Big], \qquad x \neq x', \tag{2}$$

$$G(y, y') = \pm\frac{1}{2} i\theta\, \epsilon_\perp (y - y') - 2\alpha' G^{-1} \log \Big[\vartheta_1 (\frac{|y - y'|}{2\tau}, \frac{i}{\tau}) / D(\tau) \Big], \qquad x = x', \tag{3}$$

where $q = e^{-\frac{\pi}{\tau}}$, $\pm$ correspond to $x = 1$ and $x = 0$ respectively, and $\epsilon_\perp(y) = sign(y) - \frac{y}{\tau}$. The open string parameters are as in [1]:

$$G = (g - 2\pi\alpha' B) g^{-1} (g + 2\pi\alpha' B) \tag{4}$$

is the open string metric, and

$$\theta = -(2\pi\alpha')^2 (g + 2\pi\alpha' B)^{-1} B (g - 2\pi\alpha' B)^{-1} \tag{5}$$

is the noncommutativity parameter. $\vartheta_{4,1}(\nu,\tau) = \vartheta_{01,10}(\nu,\tau)$ are Jacobi theta functions, while $D(\tau) = \tau^{-1}[\eta(\frac{i}{\tau})]^3$, where η is the Dedekind eta function [17].

With this propagator and the suitable modular measure, the amplitude for the insertion of N tachyonic vertex operators at $x = 1$ and $M - N$ at $x = 0$ has been calculated [14]

$$
\begin{aligned}
A_{N.M} = {} & \mathcal{N}(2\pi)^d (\alpha')^{\Delta} {G_s}^M \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{d}{2}} [\eta(i\tau)]^{2-d} q^{\frac{1}{2}\alpha' K g^{-1} K} \\
& \times \Big(\prod_{i=1}^{M} \int_0^{y_{i-1}} dy_i \Big) \prod_{i=1}^{N} \prod_{j=N+1}^{M} \left[q^{\frac{1}{4}} \vartheta_4 (\frac{|y_i - y_j|}{2\tau}, \frac{i}{\tau}) / D(\tau) \right]^{2\alpha' k_i G^{-1} k_j} \\
& \times \prod_{i<j=1}^{N} e^{-\frac{1}{2} i \epsilon_\perp (y_i - y_j) k_i \theta k_j} \left[\vartheta_1 (\frac{|y_i - y_j|}{2\tau}, \frac{i}{\tau}) / D(\tau) \right]^{2\alpha' k_i G^{-1} k_j} \\
& \times \prod_{i<j=N+1}^{M} e^{-\frac{1}{2} i \epsilon_\perp (y_i - y_j) k_i \theta k_j} \left[\vartheta_1 (\frac{|y_i - y_j|}{2\tau}, \frac{i}{\tau}) / D(\tau) \right]^{2\alpha' k_i G^{-1} k_j} + noncycl.\, perm.
\end{aligned}
\tag{6}
$$

Here $\mathcal{N}$ is the normalization constant, $d = p + 1$, $\Delta = M\frac{d-2}{4} - \frac{d}{2}$, G_s is the open string coupling constant, $K = \sum_{i=1}^{N} k_i$ is the sum of all momenta associated with the vertex operators on the $x = 1$ boundary, and $y_0 = 2\tau$. One can see that this amplitude corresponds to nonplanar graphs: this is most easily realized when mapping the cilinder into an annulus whose internal boundary corresponds, say, at $x = 0$ and the external one at $x = 1$. We have also omitted in the amplitude the global delta function due to momentum conservation and the traces of the Chan-Paton matrices.

We want to discuss the case of $N = 1$, $M = 2$, that is a diagram with two vertex inserted on opposite boundaries, which, in the field theory limit, will become the nonplanar contribution to the two-point function. Nonplanar diagrams are the ones in which the dependence on noncommutativity does not factor out the loop integral and therefore they are substantially different from their commutative counterpart [18]; they are indeed the diagrams we are interested in. We rescale $t = 2\pi\alpha'\tau$ and $\nu_{1,2} = y_{1,2}/2\tau$. After this we set $\nu_2 = 0$ to fix the residual invariance. The result we obtain is

$$
A_{1.2} = \mathcal{N} {G_s}^2\, 2^{\frac{3d}{2}} \pi^{\frac{3d}{2}-2} {\alpha'}^{\frac{d}{2}-3} \int_0^\infty dt\, t^{1-\frac{d}{2}} \left[\eta (\frac{it}{2\pi\alpha'}) \right]^{2-d} \times
$$

$$e^{-\frac{\pi^2\alpha'^2}{t}kg^{-1}k}\int_0^1 d\nu\left[\frac{e^{-\frac{\pi^2\alpha'}{2t}}\vartheta_4(\nu,\frac{2\pi i\alpha'}{t})}{\frac{2\pi\alpha'}{t}\left[\eta(\frac{2\pi i\alpha'}{t})\right]^3}\right]^{-2\alpha' kG^{-1}k}, \tag{7}$$

k being the external momentum.

Now we perform the zero-slope limit $\alpha' \to 0$ keeping t, ν, θ and G fixed: this can be done setting $\alpha' \sim \epsilon^{\frac{1}{2}}$ and the closed string metric $g \sim \epsilon$, and then sending $\epsilon \to 0$ [1]. The formulae for the asymptotic values of the functions in the integrand are as follows: from the expression $\eta(s) = x^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-x^m)$ where $x = \exp[2\pi i s]$, we deduce the behaviour

$$\eta(\frac{it}{2\pi\alpha'}) \sim \exp[-\frac{t}{24\alpha'}];$$

by using the property that $\eta(s) = (-is)^{-\frac{1}{2}}\eta(-\frac{1}{s})$ we get

$$\eta(\frac{2\pi i\alpha'}{t}) \sim (\frac{2\pi\alpha'}{t})^{-\frac{1}{2}}\exp[-\frac{t}{24\alpha'}],$$

and finally, since one has the property $\vartheta_4(\nu,\tau) = (-i\tau)^{-\frac{1}{2}}\exp[-\frac{i\pi\nu^2}{\tau}]\,\vartheta_2(\frac{\nu}{\tau},-\frac{1}{\tau})$, and $\vartheta_2(\rho,\sigma) = \sum_{n=-\infty}^{\infty} e^{\pi i\sigma(n+\frac{1}{2})^2+2\pi i\rho(n+\frac{1}{2})}$, we obtain

$$\theta_4(\nu,\frac{2\pi i\alpha'}{t}) \sim (\frac{2\pi\alpha'}{t})^{-\frac{1}{2}} e^{\frac{t}{2\pi\alpha'}[-\frac{\pi}{4}+\pi(\nu-\nu^2)]}.$$

The field theory coupling constant is related to the open string parameters as $g_f \sim G_s\,\alpha'^{\frac{d-6}{4}}$ and the mass is $m^2 = \frac{2-d}{24\alpha'}$. Furthermore $g^{-1} = -\frac{1}{(2\pi\alpha')^2}\theta G\theta$. Putting everything together, one obtains [14]

$$A_{1.2} \sim {g_f}^2\int_0^{\infty} dt\, t^{1-\frac{d}{2}}\, e^{-m^2 t + k\theta G\theta k/4t}\int_0^1 d\nu\, e^{-t\,\nu(1-\nu)\,kG^{-1}k}, \tag{8}$$

where some unimportant constants have been omitted. This is exactly the expression for the two-point function in the noncommutative ϕ^3 theory, and we see that the dimension of the brane worldvolume may serve as a parameter of dimensional regularization for the effective field theory. An important point is that now we choose the brane to be actually a string, therefore $d = 2$. This case is peculiar since the tachyon mass squared goes to zero from below (if $d > 2$ the effective theory is naturally tachyonic, $m^2 = \frac{2-d}{24\alpha'}$, we recall that this represents a two-tachyon scattering in string theory).

In this situation the effective field theory limit (8) is the nonplanar amplitude of a massless scalar in two-dimensional ϕ^3 theory

$$A_{1.2}(d=2) \sim {g_f}^2 \int_0^\infty dt \int_0^1 d\nu \, e^{-t\,\nu\,(1-\nu)\,kG^{-1}k + \frac{1}{4t}k\theta G\theta k}. \tag{9}$$

In two dimensions θ is proportional to the antisymmetric tensor: $\theta_{\mu\nu} = \theta\epsilon_{\mu\nu}$. We are therefore in the right situation to study the effects of electric-type backgrounds. The open string metric is proportional to the flat Minkowski metric $\eta_{\mu\nu}$, and we define a constant G such that $G^{-1}k^2 = k^\mu G^{-1}_{\mu\nu}k^\nu$, where k^2 is the usual Minkowski invariant. The field theory amplitude can then be written as

$$A_{1.2}(d=2) \sim {g_f}^2 \int_0^\infty dt \int_0^1 d\nu \, e^{-k^2[G^{-1}t\,\nu(1-\nu) - \frac{\theta^2}{4G^{-1}t}]}. \tag{10}$$

Already in the commutative case such massless scalar theories have severe infrared problems. Therefore we add a small positive mass as a regulator and get

$$A_{1.2}(d=2) \sim {g_f}^2 \int_0^\infty dt \int_0^1 d\nu \, e^{-{m_0}^2 t - k^2[G^{-1}t\,\nu\,(1-\nu) - \frac{\theta^2}{4G^{-1}t}]}. \tag{11}$$

Setting for the moment $G = 1$ for simplicity, we see that the integral in (11) is convergent in the strip $\{-4{m_0}^2 < \Re k^2 < 0\}$

$$A_{1.2}(d=2) \sim \int_0^1 d\nu \Big[-\theta^2 k^2 \frac{K_1\Big(\sqrt{-\theta^2 k^2({m_0}^2 + k^2\nu(1-\nu))}\Big)}{\sqrt{-\theta^2 k^2({m_0}^2 + k^2\nu(1-\nu))}} \Big], \tag{12}$$

and after analytic continuation it defines an analytic functions with two branch cuts $\{\Re k^2 < -4{m_0}^2\}$ and $\{\Re k^2 > 0\}$. One of these is necessarily tachyonic and cutting rules are invalidated. If we send the regulator to zero, the two branch cuts get closer and closer and eventually coalesce.

Let's analize now the full string theory diagram. The complete string amplitude, before performing the field theory limit, in the case of $d = 2$, is easily found from (7) to be

$$A_{1.2}(d=2) = \mathcal{N}{G_s}^2 \frac{8\pi}{\alpha'^2} \int_0^\infty dt \int_0^1 d\nu \, e^{-\frac{\pi^2\alpha'^2}{t}kg^{-1}k} \left[\frac{e^{-\frac{\pi^2\alpha'}{2t}}\vartheta_4(\nu, \frac{2\pi i\alpha'}{t})}{\frac{2\pi\alpha'}{t}[\eta(\frac{2\pi i\alpha'}{t})]^3} \right]^{-2\alpha' kG^{-1}k} \tag{13}$$

We set consistently $g^{\mu\nu} = g\eta^{\mu\nu}$ and $B^{\mu\nu} = B\epsilon^{\mu\nu}$. A simple calculation shows that $G^{\mu\nu} = \frac{1}{g}[g^2 - (2\pi\alpha' B)^2]\eta^{\mu\nu}$. We can therefore rewrite the string amplitude (omitting some constants) as

$$A_{1.2}(d=2) = \int_0^\infty dt \int_0^1 d\nu \exp\Big(-k^2\Big[\frac{\pi^2\alpha'^2}{g\,t} + \frac{2\alpha' g}{(g^2-(2\pi\alpha' B)^2)}\log\Big[\frac{e^{-\frac{\pi^2\alpha'}{2t}}\vartheta_4(\nu,\frac{2\pi i\alpha'}{t})}{\frac{2\pi\alpha'}{t}[\eta(\frac{2\pi i\alpha'}{t})]^3}\Big]\Big]\Big). \quad (14)$$

There are two regions we have to study in order to evaluate the convergence of the integral over t in (14), namely the ones close to the two extremes. We use again the asymptotic expressions we have reported above for the elliptic functions. At small t, the integrand behaves as

$$\exp\Big(-k^2\frac{\pi^2\alpha'^2}{g\,t}\Big); \quad (15)$$

therefore we have convergence for $g\Re k^2 > 0$ and a branch cut along the opposite axis. At large t the argument of the log behaves like $e^{\frac{t\nu(1-\nu)}{2\alpha'}}$; therefore the log actually produces a linear term that overcomes the first one inside the square brackets of (14), producing as a net result

$$\exp\Big(-k^2 t\frac{g}{(g^2-(2\pi\alpha' B)^2)}\nu(1-\nu)\Big). \quad (16)$$

We see that the branch cut is parameterized by the quantity $\frac{g}{(g^2-(2\pi\alpha' B)^2)} = \frac{1}{g}\frac{1}{1-\tilde{E}^2}$ where we have defined the effective electric field $\tilde{E} = \frac{2\pi\alpha' B}{g}$. This is exactly the ratio $\frac{E}{E_{cr}}$ of [6] which discriminates the stability of the string in this background: $1-\tilde{E}^2$ must be positive in order to avoid tachyonic instability. This condition is precisely what we find from cutting rules: if this parameter is positive, the region of convergency is $g\Re k^2 > 0$, and the branch cut is superimposed to the one coming from the small t analysis; therefore the total amplitude exhibits a single "physical" branch cut, which is interpreted in terms of intermediate states belonging to the string spectrum. If, on the contrary, this parameter becomes negative, as it does in the Seiberg-Witten limit because $g^2 \sim \epsilon^2$ and $\alpha'^2 \sim \epsilon$, a new branch cut appears on the opposite axis, which is just the unphysical one we find in the effective field theory.

III. CONCLUSIONS

We have shown that the breakdown of perturbative unitarity in noncommutative electric-type field theories can be related, from the point of view of cutting rules, to the appearance of a tachyonic branch cut in the corresponding string theory amplitude when the electric field overcomes the critical value. We have analyzed a simple case of a two dimensional brane-worldvolume which is effectively described by a massless scalar ϕ^3 theory. The string amplitude below the critical field is perfectly unitary, but the zero-slope limit forces the electric field to overcome its critical value. At the same time the quantity that parameterizes the branch cut becomes negative, and the amplitude enters the region of instability. The corresponding noncommutative field theory violates unitarity in this situation.

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